Lecture: 1-4: Exponential Functions

Example 1: Suppose you are offered a job that lasts one month. Which of the following methods of payment do you prefer?

(a) One million dollars at the end of the month.

total pay: \$1,000,000

(b) One cent the first day, two cents the second, four cents the third, etc.

day 1: 1¢
day 2: 2¢ = 2^1
day 3: 4¢ = 2^2
day 4: 8¢ = 2^3

etc… day 28 is 2^{27} cents
or… 134217728 ¢
\$ 1,342,177.28 on day 28

Laws of Exponents If a and b are positive numbers and x and y are real numbers, then

(a) $b^x b^y = \underline{b^{x+y}}$

$x^2 x^3 = xx \cdot xxx = x^5$

(b) $\frac{b^x}{b^y} = \underline{b^{x-y}}$

$\frac{x^5}{x^2} = \frac{xxxxx}{xx} = x^3$

(c) $(b^x)^y = \underline{b^{xy}}$

$(x^2)^3 = x^2 \cdot x^2 \cdot x^2 = x^6$

(d) $(ab)^x = \underline{a^x b^x}$

$(2x)^3 = 2x \cdot 2x \cdot 2x$
$= 8x^3$

Example 2: Use the laws of exponents to simplify the following expressions.

(a) $e^2 e^x = \boxed{e^{2+x}}$

not e^{2x}

(b) $(e^{5x})^2 = e^{5x \cdot 2} = \boxed{e^{10x}}$

NOT e^{25x^2}

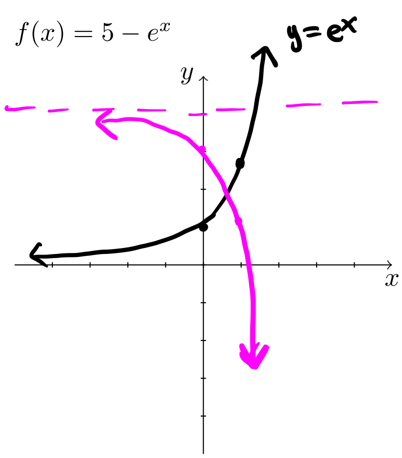
(c) $\frac{5^2}{5^x} = \boxed{5^{2-x}}$

not $5^{2/x}$ or 1^{2-x}

Example 3: Graph the following exponential functions.

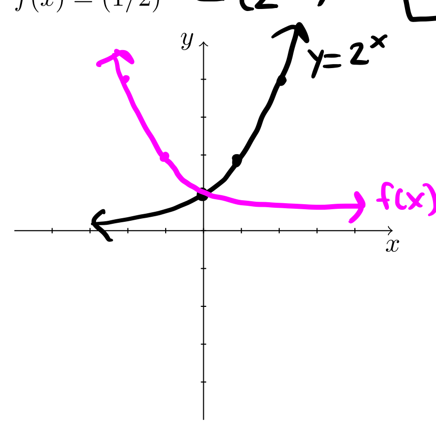
(a) $f(x) = 5 - e^x$

reflect $y = e^x$ over the x-axis and move up 5.



(b) $f(x) = (1/2)^x = (2^{-1})^x = \boxed{2^{-x}}$

reflect $y = 2^x$ over the y-axis..



x	y=2^x
0	1
1	2
2	4

Example 4: Find the exponential function $f(x) = a \cdot b^x$ who passes through the points $(1, 6)$ and $(3, 24)$.

x y   x y

$(1,6) \Rightarrow 6 = a \cdot b^1 \Rightarrow a = \boxed{6/b}$ (input this)

$(3,24) \Rightarrow 24 = a \cdot b^3$

$24 = 6/b \cdot b^3$

$24 = 6b^2$

$4 = b^2$

$b = 2$ (b can't be -2)

$a = 6/b = 6/2 = 3$

$\boxed{f(x) = 3 \cdot 2^x}$ ← your final answer is your function.

Example 5: The half-life of strontium-90 is 25 years, meaning half of any given quantity of strontium-90 will disintegrate in 25 years.

(a) If a sample of strontium-90 has a mass of 100 mg, find an expression for the mass $m(t)$ that remains after t years.

$$\boxed{m(t) = 100 \cdot (1/2)^{(t/25)}}$$

t	$m(t)$
0	100
25	50 $(100 \cdot 1/2)$
50	25 $(100 \cdot 1/2 \cdot 1/2)$
75	12.5
100	6.25
125	3.125

(b) Find the mass remaning after 40 and 80 years.

$m(40) = 100(1/2)^{(40/25)} \approx \boxed{32.988}$; $m(80) = 100(1/2)^{(80/25)} \approx \boxed{10.882}$

(c) Estimate the time required for the mass to be reduced to 5 mg.

between 100 + 125 years... need to solve $5 = 100(1/2)^{t/25}$ for t.

Inverse Functions

Generally speaking, inverse functions are functions that "undo" one another. For example, if I square a number, to undo this operation I take a square root. Thus, $f(x) = x^2$ and $g(x) = \sqrt{x}$ are inverse functions. There are some technicalities to this relationship, but the basic idea that inverses "undo" each other is a good place to start.

if $f(x) = x+2$, $f^{-1}(x) = x-2$

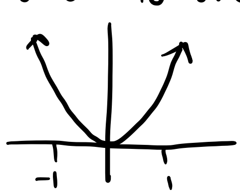
if $f(x) = x/2$, $f^{-1}(x) = 2x$

if $f(x) = x^3$, $f^{-1}(x) = \sqrt[3]{x}$

Defintion: A function f is called **one-to one** if it never takes on the same values twice. That is,

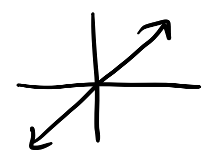
$$f(x_1) \neq f(x_2) \quad \text{whenever} \quad x_1 \neq x_2$$

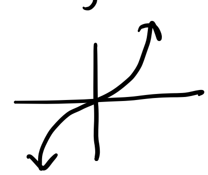
not one-to-one $f(x) = x^2$



has the same y-value at $x = \pm 1$ (and more)

one to one $f(x) = x$, x^3



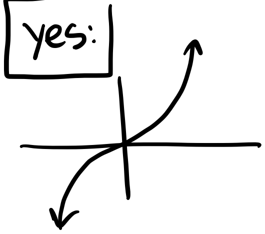


Horizontal Line Test A function f is one-to-one if and only if no horizontal line intersects its graph more than once.

Example 6: Are the following functions one-to-one?

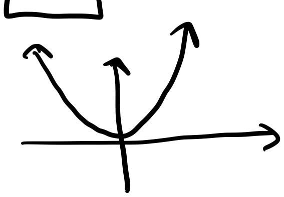
(a) $f(x) = x^3$

$\boxed{\text{yes.}}$



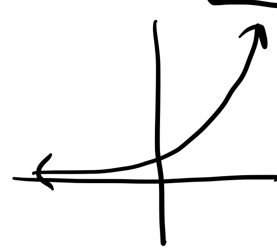
(b) $f(x) = x^2$

$\boxed{\text{no}}$



(c) $f(x) = e^x$

$\boxed{\text{Yes}}$



Definition: Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A. It is defined by

$$f^{-1}(y) = x \text{ if and only if } f(x) = y$$

for any y in B.

One consequence of the definition above are the following cancellation equations. $f(x) = x^3$, $f^{-1}(x) = \sqrt[3]{x}$

- $f(f^{-1}(x)) = x$ for all x in B ex: $f(f^{-1}(x)) = f(\sqrt[3]{x}) = (\sqrt[3]{x})^3 = x$
- $f^{-1}(f(x)) = x$ for all x in A ex: $f^{-1}(f(x)) = f(x^3) = \sqrt[3]{(x^3)} = x$

note f & f^{-1} cancel in some sense!

Example 7: If $f(1) = 5$, $f(3) = 7$, and $f(8) = -10$ find the following.

(a) $\boxed{f^{-1}(7) = 3}$ what x gives us a y-value of 7?

Q: $f(x) = 7$?

(b) $\boxed{f^{-1}(5) = 1}$

Example 8: Find the inverse of the following functions. Give the domain and range of the inverse.

(a) $f(x) = (x + 2)^3 - 5$

① $y = (x+2)^3 - 5$

② solve for x → this forces you to undo stuff in the "right" order.

$y + 5 = (x+2)^3$

$\sqrt[3]{y+5} = x + 2$

$\sqrt[3]{y+5} - 2 = x$

③ $\boxed{f^{-1}(x) = \sqrt[3]{x+5} - 2}$

(b) $f(x) = \dfrac{2x + 3}{x - 5}$ [D: $x \neq 5$, R: $y \neq 2$]

$y = \dfrac{2x+3}{x-5}$

$y(x-5) = 2x+3$

$xy - 5y = 2x + 3$ ← get x's on the same side

$xy - 2x = 3 + 5y$

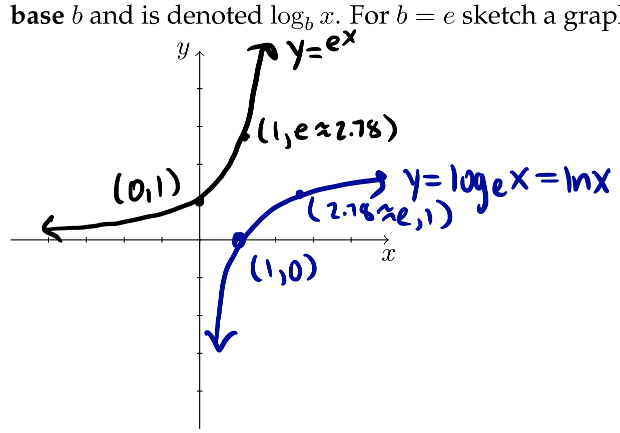
$x(y-2) = 3 + 5y$ ← factor out x

$x = \dfrac{3+5y}{y-2}$

$\boxed{f^{-1}(x) = \dfrac{3+5x}{x-2}}$ $\boxed{\begin{array}{l} D: x \neq 2 \\ R: y \neq 5 \end{array}}$

Logarithmic Functions

If $b \neq 1$, the exponential function $f(x) = b^x$ is either increasing or decreasing is therefore one-to-one by the Horizontal Line Test. Thus, this function has an inverse function which we call the **logarithmic function with base** b and is denoted $\log_b x$. For $b = e$ sketch a graph of $f(x) = e^x$ and $f^{-1}(x) = \log_e x = \ln x$.



Observations:

$f(x) = e^x$ has D: $(-\infty, \infty)$, R: $(0, \infty)$

$f(x) = \ln x$ has D: $(0, \infty)$ R $(-\infty, \infty)$

this means $\ln x$ can't do negatives (or zero) as inputs, but it can give negatives as outputs

As the functions $f(x) = b^x$ and $g(x) = \log_b x$ are inverses, we have the cancellation equations. ($\log_e x = \ln x$.)

a) $f(g(x)) = f(\log_b x) = b^{\log_b x} = x$ for every $x > 0$

b) $g(f(x)) = f(b^x) = \log_b b^x = x$ for every x in $\mathbb{R}$

Example 10: Find the exact values of the following expressions.

a) $\log_5 125 = \log_5 5^3 = \boxed{3}$ b) $\ln e^5 = \boxed{5}$ c) $\boxed{\ln \frac{1}{e^2} = -2}$

$5^? = 125$ ans: 3 $e^? = e^5$ $e^? = 1/e^2$

Laws of Logarithms If x and y are positive numbers, then

1. $\log_b(xy) = \log_b x + \log_b y$ → why: $\log_b(xy) = \log_b(b^{\log_b x} b^{\log_b y})$
2. $\log_b(x/y) = \log_b x - \log_b y$ → simliar to 1 $= \log_b(b^{\log_b x + \log_b y})$
3. $\log_b(x^r) = r \log_b x$ → $\log_b x^4 = \log_b(x x x x)$ $= \log_b x + \log_b y$

$= \log_b x \ \log_b x + \log_b x + \log_b x$

$= 4 \log_b x$

Example 11: Use properties of logarithms to express the following quantities as one logarithm (a) and expand the logarithm in (b).

(a) $\log b + 2\log c - 3\log d = \log b + \log c^2 - \log d^3$

$= \log(bc^2) - \log d^3$

$= \boxed{\log\left(\frac{bc^2}{d^3}\right)}$ ← this is not the same as $\frac{\log(bc^2)}{\log d^3}$, which is wrong.

(b) $\ln\left(\frac{\sqrt{x^2+5}(x-3)^5}{(x+5)^2}\right) = \ln((x^2+5)^{1/2}) + \ln(x-3)^5 - \ln(x+5)^2$

$= \boxed{\frac{1}{2}\ln(x^2+5) + 5\ln(x-3) - 2\ln(x+5)}$

there is no rule for $\ln(x^2+5)$ or $\ln(A+B)$. It simply cannot simplify more.

Example 12: Solve the following equations for x.

(a) $\ln(x+5) - 1 = 7$

$\ln(x+5) = 8$

$e^{\ln(x+5)} = e^8$ (exponentiate!)

$x+5 = e^8$

$\boxed{x = e^8 - 5}$ ← note, WebAssign will want exact answers like this. No decimals unless it asks!

(b) $e^{2x-5} + 4 = 10$

$e^{2x-5} = 6$ (log both sides!)

$\ln(e^{2x-5}) = \ln 6$

$2x - 5 = \ln 6$

$2x = \ln 6 + 5$

$\boxed{x = \frac{1}{2}\ln 6 + \frac{5}{2}}$

Example 13: Find the domain of the following functions.

(a) $f(x) = \dfrac{1-e^{x^2}}{1-e^{1-x^2}}$ denom can't be zero

$1 - e^{1-x^2} = 0 \Rightarrow 1 = e^{1-x^2}$

$\Rightarrow \ln 1 = 1 - x^2$

$\Rightarrow 0 = 1 - x^2$

$\Rightarrow x^2 = 1 \Rightarrow x = \pm 1$ leave these out

$\boxed{\text{so D: } (-\infty,-1) \cup (-1,1) \cup (1,\infty)}$

(b) $g(x) = \sqrt{e^x - 2}$ must be positive

$e^x - 2 \geq 0$

$e^x \geq 2$

$x \geq \ln 2$

$\boxed{\text{D: } [\ln 2, \infty)}$

Common Mistakes and Misconceptions

Example 14: Are the following statements true or false? If either case, explain why. If possible, change the false statements so that they are a true statement.

(a) $(a+b)^2 = a^2 + b^2$

False!

$(a+b)^2 = (a+b)(a+b)$
$= a^2 + ab + ba + b^2$
$= a^2 + 2ab + b^2$

In words: exponents do not distribute over addition!

(b) $\sqrt{x^2+4} = x+2$

False!

check $x = 1 \rightarrow \sqrt{1^2+4} = 1+2$?
$\sqrt{5} = 1+2$? NO!

This thing just doesn't simplify.

(c) $\dfrac{a+b}{c+d} = \dfrac{a}{c} + \dfrac{b}{d}$

False!

is $\dfrac{1+2}{2+2} = \dfrac{1}{2} + \dfrac{2}{2} \leftarrow \dfrac{3}{2}$

$\uparrow$ $3/4$ Absurd!

(d) $\dfrac{a+b}{c} = \dfrac{a}{c} + \dfrac{b}{c}$

This is true! Common denom $\rightarrow$ add numerators

(e) $\ln(x+y) = \ln x + \ln y$

False!

Try say $x = y = 1$
$\ln(1+1) = \ln(1) + \ln(1)$?
$\ln 2 = 0 + 0$ NO

(f) $\dfrac{\ln x}{\ln y} = \ln\left(\dfrac{x}{y}\right) \leftarrow = \ln x - \ln y$

False

is that

Try $e = x = y$
$\dfrac{\ln e}{\ln e} = \ln\left(\dfrac{e}{e}\right)$
$1/1 = \ln(1)$
$1 = 0 \leftarrow$ nope!

(g) $\ln(x-y) = \ln\left(\dfrac{x}{y}\right)$

False. Try $x = y = e \rightarrow \ln(e-e) = \ln(e/e)$
$\ln(0) = \ln 1$
$\uparrow$ undef. $= 0$ NO!

(h) $f^{-1}(x) = \frac{1}{f(x)}$

False! $f^{-1}(x)$ means "inverse function" not $1/f(x)$

$1/f(x) = (f(x))^{-1}$.

(i) $f^2(x) = (f(x))^2$

True! ex: $\sin^2 x = (\sin x)^2$